Acknowledgment

This research is partially supported by the Youth Aeronautical Science Foundation of China.

References

[1]Lupash, L. O., "A New Algorithm for Computation of the Geodetic Coordinates as a Function of Earth-Centered Earth-Fixed Coordinates," *Journal of Guidance, Control, and Dynamics,* Vol. 8, No. 6, 1985, pp. 787-789.

[2]Nautiyal, A., "Algorithm to Generate Geodetic Coordinates from Earth-Centered Earth-Fixed Coordinates," *Journal of Guidance, Control, and Dynamics,* Vol. 11, No. 3, 1988, pp. 281-283.

[3]Olson, D. K., "Calculation of Geodetic Coordinates from Earth-Centered Earth-Fixed Coordinates," *Journal of Guidance, Control, and Dynamics,* Vol. 11, No. 2, 1988, pp. 188-190.

[4]Levin, J. zev, "A Rational Parametric Approach to Latitude, Longitude, and Altitude," *Navigation: Journal of The Institute of Navigation,* Vol. 35, No. 3, 1988, pp. 361-370.

[5]Jones, B. W., *An Introduction to Modern Algebra,* Macmillan, New York, 1975, pp. 301-311.

Equation for Optimal Power-Limited Spacecraft Trajectories

John E. Prussing*

University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

Introduction

FOR a power-limited spacecraft the necessary conditions for an optimal (minimum-propellant) trajectory and the equations of motion can be combined into a single fourth-order differential equation for the position vector. Every solution to this equation then represents an optimal trajectory through the specified gravitational field. By imposing the appropriate boundary conditions on the solution of this equation, the desired optimal trajectory can be determined. A significant increase in payload can be attained by an optimal trajectory.

Cost Functional for an Optimal Trajectory

A brief derivation of the cost functional for an optimal power-limited spacecraft trajectory is given here for completeness, but is similar to others in the literature.[1] First consider the equations of motion:

$$\dot{\boldsymbol{r}} = \boldsymbol{v} \tag{1}$$

$$\dot{\boldsymbol{v}} = \boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{\Gamma} \tag{2}$$

$$\dot{m} = -b \tag{3}$$

where $\boldsymbol{r}$ and $\boldsymbol{v}$ are the position and absolute velocity vectors of the spacecraft, $\boldsymbol{g}$ is the gravitational acceleration, and $\boldsymbol{\Gamma}$ is the thrust acceleration of the engine. The mass is denoted by m, and b is the (nonnegative) mass flow rate of the engine. The thrust acceleration magnitude is $\Gamma = bc/m$, where c is the exhaust velocity.

The limit on power available to the engine from a separate power source is usually prescribed as an upper bound on the exhaust power:

$$P = \tfrac{1}{2} bc^2 \leq P_{\max} \tag{4}$$

To minimize the mass of the propellant consumed, it is equivalent to maximize the final mass for a given initial mass. Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{m}\right) = -\frac{\dot{m}}{m^2} = \frac{b}{m^2} = \frac{\Gamma^2}{2P} \tag{5}$$

where Eq. (4) has been used to eliminate the mass variable from the right-hand side of Eq. (5). Integrating Eq. (5) yields

$$\frac{1}{m_f} - \frac{1}{m_o} = \tfrac{1}{2}\int_{t_o}^{t_f} \frac{\Gamma^2(t)}{P(t)}\,\mathrm{d}t \tag{6}$$

One can maximize m_f by minimizing the right-hand side of Eq. (6). Because the solution to the equations of motion (1) and (2) depends only on the vector $\boldsymbol{\Gamma}(t)$ and the initial conditions $\boldsymbol{r}(t_o)$ and $\boldsymbol{v}(t_o)$, the right-hand side of Eq. (6) is minimized by choosing the power to be constant at its maximum value of $P(t) \equiv P_{\max}$. For this reason the cost functional to be minimized can be taken to be, simply

$$J = \tfrac{1}{2}\int_{t_o}^{t_f} \Gamma^2(t)\,\mathrm{d}t \tag{7}$$

Necessary Conditions for an Optimum

To apply the necessary conditions for a minimum of J, one forms the Hamiltonian function[2]

$$H = \tfrac{1}{2}\Gamma^2 + \lambda_r^T \boldsymbol{V} + \lambda_v^T(\boldsymbol{g} + \Gamma \boldsymbol{u}) \tag{8}$$

where $\boldsymbol{\Gamma} = \Gamma \boldsymbol{u}$ with $\boldsymbol{u}$ being a unit vector in the thrust direction.

The adjoint vectors satisfy the equations

$$\dot{\lambda}_r^T = -\frac{\partial H}{\partial \boldsymbol{r}} = -\lambda_v^T G \tag{9}$$

and

$$\dot{\lambda}_v^T = -\frac{\partial H}{\partial \boldsymbol{v}} = -\lambda_r^T \tag{10}$$

where G is the (symmetric) gravity gradient matrix $\partial \boldsymbol{g}/\partial \boldsymbol{r}$ and the boundary conditions of Eqs. (9) and (10) depend on the terminal conditions for the trajectory. Combining Eqs. (9) and (10) gives

$$\ddot{\lambda}_v^T = G(\boldsymbol{r})\lambda_v \tag{11}$$

Invoking the Minimum Principle, one minimizes the instantaneous value of the Hamiltonian H with respect to the thrust direction $\boldsymbol{u}$ by aligning $\boldsymbol{u}$ opposite to the adjoint vector λ_v. For this reason it is customary to define the *primer vector* after Lawden,[3] $\boldsymbol{p}(t) \equiv -\lambda_v(t)$, which defines the optimal thrust direction. Equation (11) can be written as the primer vector equation

$$\ddot{\boldsymbol{p}} = G(\boldsymbol{r})\boldsymbol{p} \tag{12}$$

From Eq. (10) the adjoint vector λ_r is then equal to $\dot{\boldsymbol{p}}$ and the Hamiltonian with the optimal thrust direction can then be expressed as

$$H = \tfrac{1}{2}\Gamma^2 + \dot{\boldsymbol{p}}^T\boldsymbol{v} - \boldsymbol{p}^T\boldsymbol{g} - \Gamma p \tag{13}$$

Received June 1, 1992; revision received July 13, 1992; accepted for publication July 27, 1992.

*Professor, Department of Aeronautical and Astronautical Engineering. Associate Fellow AIAA.

The Hamiltonian of Eq. (13) is then minimized with respect to the thrust acceleration magnitude Γ by setting

$$\frac{\partial H}{\partial \Gamma} = \Gamma - p = 0; \qquad \Gamma = p \tag{14}$$

which, along with

$$\frac{\partial^2 H}{\partial \Gamma^2} = 1 > 0 \tag{15}$$

indicates that a minimum of H is achieved. The optimal thrust acceleration vector $\boldsymbol{\Gamma}(t)$ then has the same direction and magnitude as the primer vector, and must therefore be equal to the primer vector

$$\boldsymbol{\Gamma}(t) = \boldsymbol{p}(t) \tag{16}$$

Equation Describing an Optimal Trajectory

Combining Eqs. (1) and (2) yields

$$\ddot{\boldsymbol{r}} = \boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{\Gamma} = \boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{p} \tag{17}$$

where Eq. (16) has been utilized. Equation (12) can be incorporated by differentiating twice

$$\boldsymbol{r}^{iv} = \ddot{\boldsymbol{g}} + \ddot{\boldsymbol{p}} = \ddot{\boldsymbol{g}} + G\boldsymbol{p} = \ddot{\boldsymbol{g}} + G(\ddot{\boldsymbol{r}} - \boldsymbol{g}) \tag{18}$$

where the superscript iv denotes the fourth time derivative and Eq. (17) has been used to eliminate the primer vector. Note that Eq. (18) is expressed completely in terms of the position vector $\boldsymbol{r}$ and its derivatives.

One next needs to evaluate the vector $\ddot{\boldsymbol{g}}$ to achieve the desired final equation. First, note that

$$\dot{\boldsymbol{g}} = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}}\dot{\boldsymbol{r}} = G\dot{\boldsymbol{r}} \tag{19}$$

Differentiating again,

$$\ddot{\boldsymbol{g}} = \dot{G}\dot{\boldsymbol{r}} + G\ddot{\boldsymbol{r}} \tag{20}$$

Substituting Eq. (20) into Eq. (18) yields

$$\boldsymbol{r}^{iv} - \dot{G}\dot{\boldsymbol{r}} + G(\boldsymbol{g} - 2\ddot{\boldsymbol{r}}) = \boldsymbol{0} \tag{21}$$

Equation (21) is the desired equation for an optimal power-limited trajectory. This is the only equation that needs to be solved to determine the optimal trajectory. Every solution of Eq. (21) for $\boldsymbol{r}(t)$ is an optimal power-limited trajectory in the gravitational field $\boldsymbol{g}(\boldsymbol{r})$. The desired solution is the one that satisfies the specified boundary conditions, as discussed above.

The optimal thrust acceleration vector can be computed once the solution $\boldsymbol{r}(t)$ is known from

$$\boldsymbol{\Gamma}(t) = \ddot{\boldsymbol{r}} - \boldsymbol{g} \tag{22}$$

If the computation in Eq. (22) is inaccurate due to the subtraction of two nearly equal vectors, then the thrust acceleration can be calculated by solving Eq. (12).

Unfortunately there is no single equation analogous to Eq. (21) for a thrust-limited engine ($0 \leq \Gamma \leq \Gamma_{\max}$). On a thrust-limited trajectory the optimal thrust direction is also the direction of the primer vector, but the thrust acceleration magnitude instantaneously switches between the limiting values of zero and $\Gamma_{\max}$ depending on the magnitude of the primer vector.

Boundary Conditions for Rendezvous and Interception

The solution to Eq. (21) satisfying specified boundary conditions, for example a rendezvous or an interception, must be determined. Because Eq. (21) is fourth-order, its solution is expressed in terms of four arbitrary constant vectors whose values must be chosen to satisfy the boundary conditions. The boundary conditions for a rendezvous dictate that the solution to Eq. (21) for $\boldsymbol{r}(t)$ satisfy the four specified values of $\boldsymbol{r}(t_o)$, $\boldsymbol{v}(t_o)$, $\boldsymbol{r}(t_f)$, and $\boldsymbol{v}(t_f)$.

For an interception, however, the final value of velocity $\boldsymbol{v}(t_f)$ is unspecified. If the terminal constraints do not depend on the final velocity, satisfaction of the necessary conditions requires a zero boundary condition on the adjoint vector associated with the velocity, $\boldsymbol{\lambda}_v(t_f)$, which is equivalent to a zero value for the primer vector. Thus, for an intercept the four boundary conditions are specified values of $\boldsymbol{r}(t_o)$, $\boldsymbol{v}(t_o)$, $\boldsymbol{r}(t_f)$, along with $\boldsymbol{p}(t_f) = 0$. From Eq. (17) it can be seen that this latter condition is a boundary condition on the final acceleration, namely that $\ddot{\boldsymbol{r}}(t_f) = \boldsymbol{g}[\boldsymbol{r}(t_f)]$.

Inverse-Square Gravity Field

The solutions to Eq. (21) for the cases of a uniform gravity field and for the Hill-Clohessy-Wiltshire linearized gravity field are developed in Ref. 4 for both rendezvous and interception. The solution for the inverse-square field will be considered here.

The inverse-square gravity field is given by

$$\boldsymbol{g}(\boldsymbol{r}) = -\frac{\mu}{r^3}\boldsymbol{r} \tag{23}$$

where μ is the gravitational constant of the central body.

The gravity gradient matrix needed in Eq. (21) is calculated to be

$$G(\boldsymbol{r}) = \frac{\mu}{r^5}(3\boldsymbol{r}\boldsymbol{r}^T - r^2 I) \tag{24}$$

where I is the 3×3 identity matrix. The matrix $\dot{G}$ in Eq. (21) is given by

$$\dot{G}(\boldsymbol{r}) = \frac{3\mu}{r^5}(\boldsymbol{r}\dot{\boldsymbol{r}}^T + \dot{\boldsymbol{r}}\boldsymbol{r}^T) - \frac{\boldsymbol{r}^T\dot{\boldsymbol{r}}}{r^2}\left[\frac{2\mu}{r^3}I + 5G(\boldsymbol{r})\right] \tag{25}$$

Substitution of Eqs. (23), (24), and (25) into Eq. (21) yields the equation for an optimal power-limited trajectory in an inverse-square gravitational field. As an example, the two-dimensional components of this equation expressed in polar coordinates are

$$\begin{aligned} r^{iv} - 6\ddot{r}\dot{\theta}^2 - 2\dot{r}\dot{\theta}\ddot{\theta} - 3r\ddot{\theta}^2 - 4r\dot{\theta}\theta^{iii} + r\dot{\theta}^4 - \frac{\mu}{r^5}(2\mu - 6r\dot{r}^2 \\ - r^3\dot{\theta}^2 + 4r^2\ddot{r}) = 0 \end{aligned} \tag{26}$$

$$r\theta^{iv} + 4r^{iii}\dot{\theta} - 4\dot{r}\dot{\theta}^3 - 6r\dot{\theta}^2\ddot{\theta} + 4\dot{r}\theta^{iii} + 6\ddot{r}\ddot{\theta} - \frac{2\mu}{r^3}(\dot{r}\dot{\theta} - r\ddot{\theta}) = 0 \tag{27}$$

where the superscripts iv and iii denote fourth and third time derivatives, respectively.

The initial conditions required to numerically integrate Eqs. (26) and (27) are the initial values of the dependent variables r and θ along with their derivatives up through third order. The values of r_o and θ_o are given by the initial position vector $\boldsymbol{r}_o$

$$[r_o \;\; \theta_o] = [r_o \;\; 0] = \boldsymbol{r}_o^T \tag{28}$$

and the values of $\dot{r}_o$ and $\dot{\theta}_o$ are obtained from the initial velocity vector $\boldsymbol{v}_o$

$$[\dot{r}_o \;\; r_o\dot{\theta}_o] = \boldsymbol{v}_o^T \tag{29}$$

The other initial conditions are designated by the constants α_1, α_2, α_3, and α_4. These are the initial components of the thrust acceleration vector and their initial rates and are the variables iterated on to achieve the desired final rendezvous or intercept conditions. In particular,

$$\ddot{r}_o = r_o\dot{\theta}_o^2 - \frac{\mu}{r_o^2} + \alpha_1 \tag{30a}$$

$$r_o\ddot{\theta}_o = \alpha_2 - 2\dot{r}_o\dot{\theta}_o \tag{30b}$$

along with

$$r_o^{iii} = \dot{r}_o\left(\frac{2\mu}{r_o^3} - 3\dot{\theta}_o^2\right) + \alpha_3 + 2\dot{\theta}_o\alpha_2 \tag{31a}$$

$$r_o\theta_o^{iii} = \frac{3\dot{r}_o}{r_o}(2\dot{r}_o\dot{\theta}_o - \alpha_2) + 2\left(\frac{\mu\dot{\theta}_o}{r_o^2} - r_o\dot{\theta}_o^3\right) - 2\dot{\theta}_o\alpha_1 + \alpha_4 \tag{31b}$$

These general initial conditions simplify for a circular initial orbit of radius r_c as follows:

$$r_o = r_c; \qquad \theta_o = 0 \tag{32}$$

$$\dot{r}_o = 0; \qquad \dot{\theta}_o = \left(\frac{\mu}{r_c^3}\right)^{1/2} \tag{33}$$

$$\ddot{r}_o = \alpha_1; \qquad r_c\ddot{\theta}_o = \alpha_2 \tag{34}$$

and

$$r_o^{iii} = \alpha_3 + 2\left(\frac{\mu}{r_c^3}\right)^{1/2}\alpha_2 \tag{35a}$$

$$r_c\theta_o^{iii} = \alpha_4 - 2\left(\frac{\mu}{r_c^3}\right)^{1/2}\alpha_1 \tag{35b}$$

In some applications fewer than four α parameters need to be iterated. One example is a time-open, angle-open, circle-to-circle low-Earth-orbit (LEO) to geosynchronous-Earth-orbit (GEO) transfer. In this case one can select $\alpha_1 = 0$ (tangential initial thrust) and choose α_2 to be a realistic value of initial thrust magnitude divided by the initial mass. The numerical integration is terminated at a time T for which GEO altitude is reached and the values of α_3 and/or α_4 are iterated to achieve zero final eccentricity. For very low thrust levels the eccentricity remains very small for the entire transfer.

Numerical Example

For an initial thrust of 4 N, mass of 1000 kg and 15 kW power, an optimal LEO to GEO transfer is obtained by integration of Eqs. (26) and (27) from $r = 1.05$ to 6.61 Earth radii with $\alpha_1 = \alpha_3 = \alpha_4 = 0$ and $\alpha_2 = 4.08 \times 10^{-4}$ in canonical units ($\mu = 1$). The transfer time T is 1440 time units, which corresponds to 13.5 days. The optimal final thrust magnitude is 2.47 N with a propellant consumption of 383 kg. By contrast, running at (non-optimal) constant 4N thrust magnitude in the optimal thrust direction takes a shorter time equal to 10.0 days, but requires a significantly larger 463 kg of propellant. Assuming this saving of 80 kg of propellant (8% of the initial mass) can be exchanged for an equal amount of payload, using the optimal trajectory can increase a 160 kg payload to 240 kg, representing an increase of 50%.

Concluding Remarks

For a power-limited spacecraft, the equations of motion and the necessary conditions for an optimal trajectory have been combined into a single fourth-order differential equation for the position vector. Every solution to this equation represents an optimal trajectory through the specified gravitational field. The equation for an arbitrary gravitational field has been derived and specialized to the inverse-square gravitational field. An example trajectory is described and the significant increase in payload attainable by an optimal trajectory is illustrated.

References

[1]Marec, J. P., *Optimal Space Trajectories,* Elsevier, New York, 1979, pp. 57–59.

[2]Bryson, A. E., Jr., and Ho, Y. C., *Applied Optimal Control,* Hemisphere, Washington, DC, 1975, Chap. 2.

[3]Lawden, D. F., *Optimal Trajectories for Space Navigation,* Butterworths, London, 1963, Chap. 3.

[4]Murphy, S. P., "A Study of Power-Limited Spacecraft Trajecto ries," M. S. Thesis, Dept. of Aeronautical and Astronautical Engineering, Univ. of Illinois at Urbana-Champaign, Urbana, IL, 1992.

Attitude Determination in Higher Dimensions

Malcolm D. Shuster*
Johns Hopkins University
Applied Physics Laboratory,
Laurel, Maryland 20723

Introduction

GENERALLY, one has little cause to estimate an attitude in spaces of dimension higher than three. This exercise, however, will afford an insight into the workings of a well-known attitude determination algorithm in three dimensions. In addition, should the dimensionality of our world ever increase without notice, we will be all the better prepared.

An $n \times n$ proper orthogonal matrix A satisfies

$$A^TA = I_{n\times n} \tag{1}$$

$$\det A = 1 \tag{2}$$

Equation (1) is equivalent to $n(n+1)/2$ constraints on the matrix A. Hence, A can have only $n(n-1)/2$ free parameters, as remarked by Bar-Itzhack[1] and Bar-Itzhack and Markley.[2] Thus, A may be represented in terms of matrices of manifestly smaller parameter dimension. For example,

$$A = \exp\{\Theta\} \tag{3}$$

where Θ is an $n \times n$ antisymmetric matrix whose independent elements are the n-dimensional generalization of the rotation vector.[1] Likewise, one may write[2]

$$A = (I + G)(I - G)^{-1} \tag{4}$$

Received April 26, 1991; revision received April 24, 1992; accepted for publication May 17, 1992.

*Senior Professional Staff, Space Department, Guidance and Control Group. Associate Fellow AIAA.